

Conceptual replication using interactive virtual scenarios from the exploratory study on understanding the Intermediate Value Theorem

Juan Gabriel Herrera Alva ¹ 

Kinrha Aguirre De la Luz ² 

Martha Leticia García Rodríguez ³ 

What problem is addressed?

According to Aguilar (2020), replication studies help reinforce confidence in the results of previous research. They help us understand how certain interventions or instructional designs work in different contexts and populations. Replication studies are an essential tool for consolidating and advancing implementation research in the field of mathematics education.

This proposal consists of an implementation and replication study, based on the results reported by Herrera et al. (in press), who analyzed the characteristics of understanding in a case-by-case proof of the Intermediate Value Theorem (IVT) using theoretical elements of understanding that led to the proposal of features of IVT understanding and were identified through the design of a hypothetical learning trajectory (HLT). The results of that study highlight students' difficulties in formulating and proving the IVT; some of these difficulties are consistent with those reported in the literature (Wasserman et al., 2022; Weber, 2002), while others were not identified in the review.

¹ gabalva@ciencias.unam.mx

Universidad Nacional Autónoma de México

² kinrha@gmail.com

Universidad Nacional Autónoma de México

³ mlgarcia@ipn.mx

Instituto Politécnico Nacional

Throughout the students' work, both signs of understanding the IVT proving and difficulties were identified simultaneously; some of these difficulties faded away, while others persisted intermittently, such as the omission and/or confusion of hypotheses and a lack of experience with strategic knowledge (Weber, 2002). A finding omitted from Herrera et al. (in press) due to space constraints was the inadequate interpretation of an auxiliary set—essential to the proof of the theorem—which, together with intermittent difficulties, led to the design of this replication study, named HLT-2, which incorporates an additional methodological component to the original HLT (hereinafter HLT-1), namely, a set of “interactive virtual scenarios with real-time feedback” (IVS). The object of the replication study is to identify patterns and difficulties in understanding the statement and proof of the Intermediate Value Theorem using HLT-2. Thus, the research question in this study is: Does HLT-2 promote understanding of the statement and proof of the IVT?

How is it addressed (theory/methods)?

HLT-2 is based on two theoretical constructs: the notion of understanding, which incorporates ideas about the connection between symbols, procedures, and geometric representations (Hiebert & Carpenter, 1992; Skemp, 1978), and the hypothetical learning trajectory construct, which includes a goal, a set of tasks, and a hypothesis about students' learning process (Simon et al., 2018). Additionally, the difficulties reported in Herrera et al. (in press) were taken into account. The HLT-2 design included: (i) the components of HLT-1, (ii) new tasks that address these difficulties, and (iii) four IVS.

The implementation of IVS is understood as a process involving multiple stakeholders, with its own foundation, structure, logic, and characteristics (Koichu et al., 2021). HLT-2 is a replication of HLT-1; according to Aguilar (2020), it allows us to understand how the new instructional design works in our case within the context of digital technology and to determine if the results from HLT-1 have improved. Currently, the part of HLT-2 that addresses Case 1 of the guided proof of the IVT has been implemented in a real classroom setting through a pilot study. The study took a qualitative approach, as the goal was to explore how students interpret each of the HLT-2 tasks through their interactions and the mediated intervention of the teacher-researchers to encourage the use of IVS at predetermined times. A case study method was used with two students enrolled in a calculus course as part of their mathematics bachelor's degree program at a major university in Mexico. The students worked in pairs and were selected because they had previously completed two activities from HLT-2. Data were collected using worksheets,

screenshots of the students' interactions with the IVS, audio recordings, notes, and semistructured interviews.

What is implemented/replicated?

Four IVS were implemented as part of HLT-2 in a replication study of HLT-1. Figure 1 illustrates an example of an IVS, where the student makes connections between the formal register (Case 1 in the IVT proof) and the geometric one, and identifies the point of contradiction (beta) by clicking on the delta neighborhood. This innovation is justified as a teaching resource grounded in theoretical elements and the results of previous research. Koichu et al. (2024) state that innovations can include research findings, theoretical frameworks, teaching materials, etc. Furthermore, its implementability—as defined by Jankvist et al. (2021), who conceive of it as how realistic and feasible it is for professionals to implement an innovation arising from the interpretation of previous research findings—is possible in a real classroom setting, as evidenced in this pilot test. The HLT-2 displays the characteristics of an innovation: (i) According to Rogers (2003), an innovation must offer a relative advantage and appeal to be perceived as superior to what it replaces. The IVS allows users to dynamically visualize and manipulate abstract concepts that are difficult to learn using static methods, such as those in HLT-1. (ii) According to Engell et al. (2021), usability and suitability are key factors for implementability. HLT-2 is intuitive and easily accessible. These features make it suitable and innovative for university settings.

Figure 1

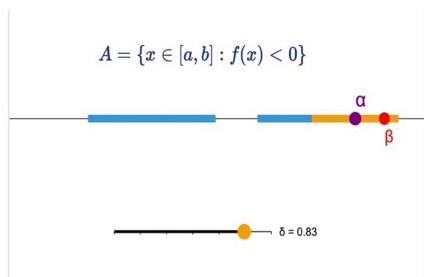
Contradiction with supremum of set A

Demostración. Sea $\alpha = \sup A$. Supongamos que $a < \alpha < b$. La idea es demostrar que $f(\alpha) = 0$.

Caso 1. Supongamos que $f(\alpha) < 0$. Observa que f satisface las hipótesis del lema de la conservación del signo, es decir, f es continua en α y $f(\alpha) < 0$; de manera que, existe una vecindad de radio δ con centro en α tal que para toda $x \in (\alpha - \delta, \alpha + \delta)$ se tiene que $f(x) < 0$.

Observación. En particular, existe $\beta \in (\alpha, \alpha + \delta)$ tal que $f(\beta) < 0$.

¿Por qué esto es una contradicción?



What is the potential importance and significance?

In the pilot test, HLT-2 showed improvements in understanding in Case 1 of the IVT proof compared to HLT-1. This is evident in a better connection between symbolic and graphical representations, in the accurate identification of the auxiliary set, and in the application of strategic knowledge at specific moments, such as the use of the sign-preservation

theorem or the definition of supremum. It is important to note that the results are not yet definitive. From the perspective of implementation research, the proposal is pedagogically sound and implementable in the classroom due to its well-planned design.

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