

Incorporating Descartes' construction of curves by continuous movement in the teaching of analytic geometry in high school

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What problem is addressed?

This work addresses the need for complementary approaches to the teaching of analytic geometry in high school, particularly in the teaching of conics. The proposal seeks to develop an intuitive or historical and evolutive approach (Moreno-Armella, 2014), in which students can learn about conics not only through drill and practice to obtain their equations but also through their geometric construction and properties underlying their generation. The proposal is intended to offer an alternative to approaches in which, following a sequence of syntactic exercises and practice, students encounter the equations of conics, which, ultimately, amount to nothing more than formulas to be memorized and applied.

A further concern is the possible discontinuity between the geometry taught at the secondary level and university geometry. In particular, geometric transformations may provide an opportunity to connect the study of conics in secondary mathematics with more advanced mathematical ideas encountered in university geometry. This concern is related to what has been described as Klein's double discontinuity and to the challenge of establishing meaningful connections between secondary and university mathematics (Wasserman et al., 2023).

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The work therefore addresses the possibility of using a historically grounded reconstruction of the emergence of analytic geometry to provide a complementary way of approaching conics in secondary mathematics, while creating opportunities to introduce geometric transformations as a bridge toward university-level geometry.

How is it addressed?

The proposal draws on a historical-evolutive perspective in mathematics education (Moreno-Armella, 2014). Following Neubrand (in Galuzzi & Neubrand, 1998), historical developments can be approached by identifying fundamental ideas and problems underlying the development of a mathematical concept. Jahnke (1995, as cited in Galuzzi & Neubrand, 1998, p. 232) similarly considers the use of history in mathematics education through the interpretation and reconstruction of historical mathematical developments.

In agreement with these perspectives, the work reconstructs a historical episode concerning Descartes' construction of curves by continuous movement, as described in his *Géométrie*. The reconstruction is implemented through Dynamic Geometry Software (DGS), from an instrumental-semiotic mediation perspective (Moreno-Armella & Sriraman, 2010). The use of DGS makes visible the invariant (and variant) relationships and geometric properties involved in the construction of the simulation of the artefact, whose movement traces the curve, allowing participants to explore how a geometrically generated curve can be related to its algebraic representation (see a diagram of this construction in Hoyos & Pecina, 2024).

An exploratory implementation was conducted through professional-development workshops with secondary mathematics teachers. A total of 49 secondary mathematics teachers participated in three separate workshops: 5 in the first workshop, 9 in the second, and 35 in the third. Each workshop consisted of a single three-hour session. The workshops were conducted at different times and locations: the first in Autlán, Jalisco; the second in Mexico City; and the third in Culiacán, Sinaloa. The workshops were structured as step-by-step guided construction activities. The sequence of steps was organized in an activity guide prepared by the teacher educator. At each step, the teacher educator demonstrated the corresponding construction on a computer connected to an LCD projector, while the participating teachers simultaneously reproduced the construction on their own computers using dynamic geometry software (*Cabri-II* or *GeoGebra*). After providing each instruction, the teacher educator circulated among the participants, observing their work and providing individualized support when difficulties arose.

At the end of each workshop, teachers were asked whether and how they would incorporate the construction of curves by continuous movement into their classroom teaching. The responses that emerged during the discussion were recorded as literally as possible in a written account prepared immediately after each workshop. In some cases, the teacher educator asked individual participants to elaborate on their responses, particularly when their participation during the activity revealed particular mathematical or technological considerations. The responses were examined through a qualitative descriptive and interpretive analysis, considering the reasons expressed by teachers in relation to the feasibility and possible adaptation of the proposal to their classrooms.

What is implemented?

The implemented innovation is a DGS-based reconstruction of Descartes' construction of curves by continuous movement. The implementation involves two crucial and interconnected uses of DGS. First, the software is used to reconstruct the Euclidean geometric device described by Descartes and to activate its continuous movement, which generates the curve. Second, DGS is used to make the geometric relationships within the moving device transparent, allowing the participants to identify invariant and varying magnitudes and to use them to derive the algebraic equation of the generated curve. In this sense, the DGS environment supports a reconstruction of the transition from a Euclidean geometric construction to an algebraic representation of the curve, echoing the historical emergence of Cartesian analytic geometry.

The implementation was explored in the three workshops described above. Participant teachers constructed the device step by step on their own computers while following the activity guide and the teacher educator's demonstrations. They then activated and manipulated the digital device, observing the movement and the resulting curve. Rather than obtaining the equation automatically from the software, participants identified the geometric magnitudes that remain invariant and those that vary during the movement. The relevant segments and magnitudes were then named and related geometrically. By applying Thales' Theorem and establishing proportional relationships among these magnitudes, the participants progressively reduced and compared these relationships until they derived an algebraic equation, that of Descartes' *hiperboloide*¹ in each workshop.

¹ In Descartes' *Géométrie*, the term "*hiperboloide*" (originally in French), appears to denote a planar curve, a higher-order geometric curve he devised to solve optical refraction problems. In other words, the term *hiperboloide* in Descartes's original text, does not refer to the three dimensional surface known in modern calculus by rotating a hyperbola.

The teachers' responses, arguments, and reasons concerning the potential adoption of the proposal in their classrooms indicated that they recognized the approach as innovative and complementary to their teaching of conics. However, they did not express a clear or enthusiastic view that the activity, as experienced in the workshop, would be feasible to reproduce directly in their classrooms. Some participants instead considered possible ways of adapting or otherwise approaching the reviewed reconstruction of curves with their students.

What is the potential importance and significance?

The potential significance of the innovation lies in offering a different meaning for the study of conics. Rather than approaching conics primarily through drill and practice to obtain their equations, the DGS reconstruction makes it possible to observe invariant geometric properties of a moving construction and to connect those properties with the corresponding algebraic equation.

The fact that the construction generates a hyperbola through a moving Euclidean device also provides an opportunity to consider curves that are not necessarily presented in the conventional position associated with the introduction of the Cartesian coordinate system at school. This creates a natural entry point to geometric transformations. Geometric transformations may therefore serve as a bridge between the treatment of analytic geometry and conics in secondary mathematics and more advanced geometry at university level.

The responses of the participating teachers suggest that the reconstruction of Descartes' construction of curves was perceived as innovative and complementary, while also revealing difficulties concerning its direct implementation in current classroom practices. These difficulties are relevant because the proposal requires students to engage simultaneously with geometric properties that may receive limited attention in conventional treatments of conics and with DGS environments for which teachers may have varying degrees of experience.

The potential significance of the proposal extends beyond the particular construction implemented in the workshops. Within a broader research trajectory, the study of geometric transformations provides a possible connection from secondary analytic geometry toward university geometry, including subsequent work involving hyperbolic geometry and/or hyperbolic tessellations. In this sense, geometric transformations emerging naturally from the study of the generated conics may constitute a mathematical bridge across educational levels.

The working hypothesis underlying the proposal is that exploring Descartes' construction together with its DGS representation may enable

students to appropriate fundamental geometric properties and use them in solving mathematical problems. Further implementation with students is needed to investigate this potential and to determine how the construction of dynamic conics may contribute to the development of geometric thinking.

References

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